Math 1320: Systems of Linear Equations in Three Variables

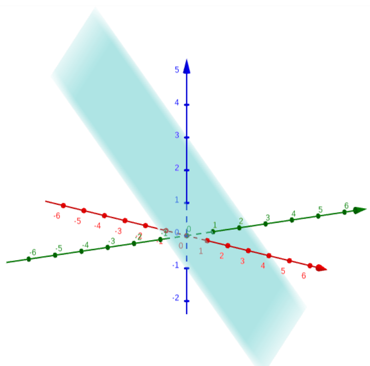
What is a linear equation in three variables? Previously, we worked with linear systems in two variables, x and y. In general, the linear equations in the systems were generally of the form:

$$Ax + By = C$$

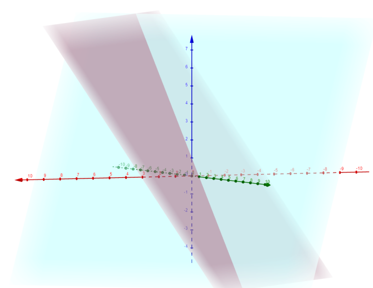
For linear equations in three variables (x, y, and z), they will generally be of the form:

$$Ax + By = Cz = D$$

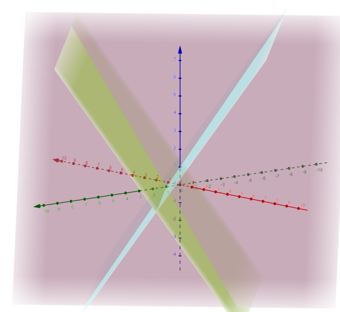
The graph of a linear equation in three variables is a plane in three-dimensional space, as shown below:



One Plane



Two Planes



Three Planes

⋆ Note: The two planes occurs when we have a system of 2 linear equations in three variables. In this case, the system will have infinitely many solutions.

How do we solve a system of linear equations in three variables? Similarly to solving linear systems in two variables, we will use an elimination method. Since we are working in a three-dimensional space, our solution will be an ordered triple, (x, y, z), that satisfies all equations in the system.

Solving Linear Systems in Three Variables by Eliminating Variables	
Step 1	Reduce the system to two equations in two variables. This is usually accomplished by taking two different pairs of equations and using the addition method to eliminate the same variable from both pairs.
Step 2	Solve the resulting system of two equations in two variables using addition or substitution. The result is an equation in one variables that gives the value of that variable.
Step 3	Back-substitute the value of the variable found in step 2 into either of the equations in two variables to find the value of the second variable.
Step 4	Use the values of the two variables from steps 2 and 3 to find the value of the third variable by back-substituting into one of the original equations.
Step 5	Check the proposed solution in each of the original equations.

It does not matter which variable we choose to eliminate first, as long as we eliminate the same variable in two different pairs of equations.

Example 1. Solve the system: $\begin{cases} x - y + 3z = 8 & \text{(Eq. 1)} \\ 3x + y - 2z = -2 & \text{(Eq. 2)} \\ 2x + 4y + z = 0 & \text{(Eq. 3)} \end{cases}$

Step 1: Reduce the system to two equations in two variables. Let's eliminate x with equations 1 and 2:

$x - y + 3z = 8$	Multiply by -3 $\longrightarrow$	$-3x + y - 9z = -24$
$3x + y - 2z = -2$	No Change $\longrightarrow$	$3x + y - 2z = -2$
	Add:	$4y - 11z = -26$ (Eq. 4)

Now eliminate x with equations 1 and 3:

$x - y + 3z = 8$	Multiply by -2 $\longrightarrow$	$-2x + 2y - 6z = -16$
$2x + 4y + z = 0$	No Change $\longrightarrow$	$2x + 4y + z = 0$
	Add:	$6y - 5z = -16$ (Eq. 5)

Now we have a new system with two equations in two variables:

$$\begin{cases} 4y - 11z = -26 & (Eq.4) \\ 6y - 5z = -16 & (Eq.5) \end{cases}$$

Step 2: Now we need to solve our new system. Let's eliminate y:

$4y - 11z = -26$	Multiply by $\frac{-2}{3}$ $\longrightarrow$	$4y - 11z = -26$
$6y - 5z = -16$	Multiply by $\frac{-2}{3}$ $\longrightarrow$	$-4y + \frac{10}{3}z = \frac{32}{3}$
	Add:	$-\frac{23}{3}z = -\frac{46}{3}$
		$z = 2$ (Multiply both sides by $-\frac{3}{23}$)

Step 3: Back-substitute 2 for z in equation 4 or equation 5 to find the value of y. Let's use equation 5:

$6y - 5z = -16$	Equation 5
$6y - 5(2) = -16$	Substitute 2 for z
$6y - 10 = -16$	Multiply
$6y = -6$	Add 10 to both sides
$y = -1$	Divide both sides by 6

Step 4: Back-substitute 2 for z and -1 for y in one of the original equations to find the value of x. Let's use equation 1:

$x - y + 3z = 8$	Equation 1
$x - (-1) + 3(2) = 8$	Substitute 2 for z and -1 for y
$x + 1 + 6 = 8$	Multiply
$x + 7 = 8$	Add
$x = 1$	Subtract 7 from both sides

The proposed solution is $(1, -1, 2)$.

Step 5: Check our solution. Does $(1, -1, 2)$ make the equations in our original system true? Yes, by substituting the values for x, y, and z into each of the three original equations, we get three true statements.

Practice Problems: Solve each system.

1. $\begin{cases} x + 2y - 3z = -3 \\ 2x - 5y + 4z = 13 \\ 5x + 4y - z = 5 \end{cases}$ [(2, -1, 1)]

2. $\begin{cases} 2x + y - z = 5 \\ 3x - 2y + z = 16 \\ 4x + 3y - 5z = 3 \end{cases}$ [(4, -1, 2)]